# UNIT II :

## Recap of State Variable analysis

There are two basic methods of control system design. Transfer function approach which forms classical method and state space or state variable approach which falls under modern control design method.

Advantages of state variable analysis (SVA):

State variable analysis is the most powerful method for Multi input Multi output (MIMO), linear/non –linear systems. Transfer function approach is too tedious for MIMO systems.

SVA gives information of the output as well as the internal state of the system

Initial conditions are incorporated in SVA whereas they are neglected in TF approach.

For large sized problems, use of computers is convenient in SVA

Flexibility in the choice of state variable is there in SVA

Some definition of SVA:

State: The state of a dynamic system is the smallest set of variables (called state variables) such that the knowledge of these variables at $t = t_0$ together with the knowledge of input for $t \geq t_0$ completely determines the behavior of the system for any time $t \geq t_0$.

State variables: State variables of a system are the smallest set of variables which determines the state of a dynamic system. They are usually indicated by $x_1, x_2, x_3, \dots, x_n$ for an $n^{th}$ order system.

State vector: If 'n' state variables are needed to completely describe the behavior of a given system, then these 'n' state variables can be considered can be considered the n components of a vector. Such a vector is called a state vector. Mathematically state vector is represented by $X = [x_1, x_2, \dots, x_n]^T$.

State space: The n-dimensional space whose coordinate axes consist of x1 axis, x 2 axis, … xn axis where $x_1, x_2, x_3, \dots, x_n$ are state variable, is called a state space.

State Model: Consider the following equations to describe any given system:

$$\dot{x}_1 = a_{11}\, x_1 + a_{12} x_2 + ... + a_{1n} x_n + b_{11}\, u_1 + b_{12} u_2 + ... + a_{1m} u_m \,;$$

$$\dot{x}_2 = a_{21}\, x_1 + a_{22} x_2 + ... + a_{2n} x_n + b_{21} u_1 + b_{22} u_2 + ... + a_{2m} u_m \,;$$

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$$\dot{x}_n = a_{n1} x_1 + a_{n2} x_2 + ... + a_{nn} x_n + b_{n1}\, u_1 + b_{n2} u_2 + ... + b_{nm} u_m \,;$$

These ‘n’ equations can be expresses in vector –matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{21} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \qquad \ldots \qquad 1.1$$

Equation 1.1 can be written as

$$\dot{X} = [A]\,\overline{X} + [B]\overline{U} \qquad \ldots \qquad 1.2$$

Equation 1.2 is called **the state equation** where

$[A]$ = ( n X n ) order is called System Matrix and

[ B] = ( n X m) order is called the input matrix.

[ X ] = ( n X 1) order is column vector of state variables, $x_1$, $x_2$, … , $x_n$

$[\dot{X}]$ = (n X 1) order is a column vector which is the derivative of state variables.

Consider the following set of equations to represent the outputs of the same given system.

$$y_1 = c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \ldots + d_{1m}u_m;$$

$$y_2 = c_{21}x_1 + c_{22}x_2 + \ldots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \ldots + d_{2m}u_m;$$

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$$y_p = c_{p1}x_1 + c_{p2}x_2 + \ldots + c_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \ldots + d_{pm}u_m;$$

These ‘p’ equations can be expresses in vector –matrix form as

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{21} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{21} & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \quad \ldots \quad 1.3$$

Equation 1.3 can be written as

$$Y = [C]\,\overline{X} + [D]\overline{U} \quad \ldots \quad 1.4$$

Equation 1. 4 is called the Output equation where

[ C ] = (n X p) order is called the output matrix

[D] = ( p X m) order is called the Transmission matrix.

[ Y] = ( p X 1) order is called output vector

[U] = (m X 1 ) order is called input vector.

Hence state model of a dynamic system consists of state equations ( 1.2) and output equations (1.4).

**Construction of state model:**

- By using physical variables
- By using phase variables
- By using Canonical variables

## Eigen Values and Eigen vectors

Consider $[A]\bar{X} = \lambda\,\bar{X}$ where $\bar{X}$ is (n X 1) state vector and λ is a scalar.

The above equation can be rewritten as $\lambda\,\bar{X} - [A]\,\bar{X} = 0$

$$[\,\lambda\,[I] - [A]\,]\bar{X} = 0 \qquad \text{... 1.5}$$

Where [ I ] is called a identity matrix.

$\left|\lambda[I] - [A]\right| = 0$ is called the characteristic equation of matrix A. Since [A] is (n X n) order, the determinant will be of n^th^ order equation of the form,

$\lambda^{n} + \alpha_1\lambda^{n-1} + \alpha_2\,\lambda^{n-2} + \ldots + \alpha_n = 0$ the n roots of the above equation $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda n$ are called the Eigen values of the matrix A.

Note: The poles of the system transfer function and the Eigen values of the state model are the same.

The non trivial solution $\bar{X} = \bar{v}_i$ of equation 1.5 is called the Eigen vector of matrix A corresponding to the Eigen value $\lambda = \lambda i$

i.e. vector $\bar{v}_i$ satisfies the following equation

$$[\,\lambda_i\,[I] - [A]\,]\,v_i = 0$$

If [A] is of n X n, then there will be 'n' number of Eigen vectors. $v_1, v_2, v_3, \ldots, v_n$ corresponding to $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda n$ respectively.

A matrix formed by placing Eigen vectors along the column is known as MODAL MATRIX.

$[\ M\ ] = \left[ v_1 \vdots v_2 \ \vdots \cdots \vdots v_n \right]$

More about eigen values:

- For a diagonal or a triangular matrix the Eigen values are given by the diagonal elements of that matrix.
- For any matrix, the sum of the Eigen values of the same matrix is equal to the TRACE of the Matrix.

**Diagonalisation of a given state model:**

Consider a system represented by a state model, with matrix A is not a diagonal matrix, as below.

$$\dot{X} = [A]\,\overline{X} + [B]\,\overline{U} \qquad \ldots. \qquad 1.6$$

$$Y = [C]\,\overline{X} + [D]\,\overline{U} \qquad \ldots. \qquad 1.7$$

Objective: To diagonalise [ A ]

## Case I Eigen values of [A] are distinct (no repeated Eigen values).

**i) [A] is a radom matrix**

**Method:** Consider the transformation

$$\overline{X} = [M]\,\overline{Z} \qquad \ldots. \qquad 1.8$$

Substituting 1.8 in 1.6, we get

$$[M]\,\dot{Z} = [A]\,[M]\,Z + [B]\,U$$

Pre-multiplying both sides by $[M]^{-1}$, we get

$$\dot{Z} = [M]^{-1}\,[A]\,[M]\,Z + [M]^{-1}\,[B]\,U \qquad \ldots. \qquad 1.9$$

Substituting 1.8 in 1.7, we get

$$Y = [C][M]Z + [D]U \qquad \ldots. \qquad 1.10$$

Equations 1.9 and 1.10 can be rewritten as,

$$\dot{Z} = [A']Z + [B']U \qquad \ldots. \qquad 1.11$$

$$Y = [C']Z + [D']U \qquad \ldots. \qquad 1.12$$

Where $[A'] = [M^{-1}][A][M]$; $\quad [B'] = [M^{-1}][B]$ ;

$[C'] = [C][M]$; $\quad [D'] = [D]$

Equations 1.11 and 1.12 represent the state model in diagonalized form, where [A'] is in diagonal or canonical form. Note that [M] is a Modal Matrix given by [ M ] = $[v_1 \vdots v_2 \vdots \cdots \vdots v_n]$ and $v_1, v_2, v_3, \ldots,$ $v_n$ are Eigen vectors corresponding to Eigen values $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda n$ respectively.

If Eigen values are distinct then, $[A'] = [M^{-1}][A][M] = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}$

ii) The method of diagonalization is [ A ] is typically in the phase variable/ companion form (as shown below) but having distinct Eigen values,

$$[A] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix}$$

Then the Modal matrix can be shown to be a special matrix called Vander Monde Matrix.

$$[V] = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \cdots & \lambda_n \\ \lambda_1^{\;2} & \lambda_2^{\;2} & \cdots & \cdots & \lambda_n^{\;2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{\;n-1} & \lambda_2^{\;n-1} & \cdots & \cdots & \lambda_n^{\;n-1} \end{bmatrix}$$

**Case II: Method of diagonalization if Eigen values are repeated in nature:**